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ON A PARADOX OF HILBERT AND BERNAYS

ABSTRACT. The paper is a discussion of a result of Hilbert and Bernays in their *Grundlagen der Mathematik*. Their interpretation of the result is similar to the standard intepretation of Tarski's Theorem. This and other interpretations are discussed and shown to be inadequate. Instead, it is argued, the result refutes certain versions of Meinongianism. In addition, it poses new problems for classical logic that are solved by dialetheism.

1. INTRODUCTION

Hilbert and Bernays' *Grundlagen der Mathematik* was published between 1934 and 1939. It contained a definitive account of the state of the art of mathematical logic at the time. Much of what it contains has been reproduced and modernised in later texts; some has simply passed into the folklore of the subject;[1] and some has been forgotten. This essay is a discussion of a small but significant passage of this kind. I shall start by giving a streamlined and slightly generalized version of a proof to be found in Hilbert and Bernays. I will then discuss what to make of it, including Hilbert and Bernays' view of the matter.[2] Because of my own views on the paradoxes of self-reference,[3] I shall be particularly concerned with its significance for dialetheism.

For orientation, consider the Liar Paradox in the setting of formal arithmetic. Given the self-referential abilities of arithmetic we can prove a fixed point theorem for formulas to the effect that for any formula of one free variable, $\varphi(x)$, there is a formula, ψ, such that $\psi \leftrightarrow \varphi(\langle\psi\rangle)$ is provable, where $\langle\psi\rangle$ is the numeral of the gödel number of ψ.[4] Now let T be a truth predicate, and let $\mathbf{\Phi}(Tx)$ be any formula containing the open sentence Tx. Then by the fixed point lemma, there is a formula, ψ, such that $\psi \leftrightarrow \mathbf{\Phi}(T\langle\psi\rangle)$. Given the substitutivity of equivalents and an instance of the T-schema, $T\langle\psi\rangle \leftrightarrow \psi$, it follows that $\psi \leftrightarrow \mathbf{\Phi}(\psi)$. In particular, take $\mathbf{\Phi}$ simply to be negation, and we have the Liar Paradox. Hilbert and Bernays' argument proceeds in essentially the same way, except that it uses a denotation function instead of a truth predicate, and identity instead of equivalence.

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2. THE FIXED POINT THEOREM

I will give Hilbert and Bernays' argument in the context of first order arithmetic, as they do. We will assume that the language contains a number of function symbols, including one for every primitive recursive function.[5] We will also assume that the denotation function for the language is represented by a function symbol; specifically, that there is a function symbol, δ, satisfying the condition that $\delta\langle s\rangle = s$, for any closed term, s. I will call this the *denotation principle*. As will be clear from the proof, a much less demanding conxent is, in fact, required for the proof: a language than can refer to its own terms, the denotation function, and some form of self-reference will suffice.

We will first demonstrate the following lemma:

LEMMA. *Let $t(x)$ be any term with one free variable, x. Then there is a closed term, s, such that:*

$$s = t(\langle s\rangle).$$

Proof. Given any term, r, let its *diagonalisation* be the term itself with each free variable replaced by $\langle r\rangle$. Diagonalisation is a primitive recursive function on codes. Let it be represented by the function symbol g. Now, consider the term $t(gx)$. Call this r. Its diagonalisation is $t(g\langle r\rangle)$. Call this s. Since the diagonalisation of r is s, $g\langle r\rangle = \langle s\rangle$. Hence, by substitutivity of identicals (si), $t(g\langle r\rangle) = t(\langle s\rangle)$. But the left hand side is exactly s. Hence we have the result. □

We can now state and prove the main result:

THEOREM. *Let $t(x)$ be any term with one free variable, x. Then there is a closed term, s, such that $s = t(s)$.*[6]

Proof. Consider the term $t(\delta x)$. By the Lemma, there is a closed term, s, such that $s = t(\delta\langle s\rangle)$. The result follows by the denotation principle and si. Note, for future reference, that the fixed point, s, is the diagonalisation of $t(\delta g x)$. □

3. HILBERT AND BERNAYS' CONCLUSION

So much for the result. What of its consequences? These arise when one applies the fixed point theorem to various functional terms. In some cases the result is entirely anodyne. For example, it is hardly news that the identity function has a fixed point. Suppose that we apply it to the successor function, however. It then establishes that there is a number that is its own successor. This is (classically) absurd. Hilbert and Bernay take it to be so, and infer that the denotation function of arithmetic cannot be represented in arithmetic. Interpreted like this, the result is an analogue of the standard interpretation of Tarski's Theorem: the truth predicate of arithmetic is hyper-arithmetic, and so cannot be represented in arithmetic.

Understood in a certain way, these interpretations of Tarski's Theorem and Hilbert and Bernays' "Theorem" are unproblematic. Tarski's Theorem shows that arithmetic with a truth predicate is inconsistent. Hence, any fragment of arithmetic that is consistent cannot represent its own truth predicate. Similarly for HB's "Theorem": since no number is equal to its successor, and this is a truth of arithmetic, the result shows that any fragment of arithmetic that can express its own denotation function is inconsistent. Hence any consistent fragment of arithmetic cannot represent its own denotation function.

More generally, however, both interpretations are problematic. Start with Tarski's Theorem. If we are constructing a language, and have specified its interpretation by fiat, then whether or not a certain notion, like truth, can be expressed in the language is a substantial question. But the argument involved in Tarski's Theorem (which is just the Liar Paradox) can be run, in a similar and obvious way, in English; and the situation here is quite different. The question of whether the notion of truth in English is expressible in English is not a substantial question: it is expressible; and this is a given. Even to raise the question makes use of the notion, and so presupposes a positive answer. *How* the phrase in question manages to attain its sense may be a difficult and unanswered question. *Whether* is has this sense is not. And given that the notion *is* expressible, together with the machinery of arithmetic, what Tarski's Theorem appears to tell us that the notion of truth is inconsistent.

The situation with HB's Theorem is similar in some ways, importantly different in others. First, the similarities. The Theorem itself is about a formal language. However, this is not of its essence, and, as with Tarski's Theorem, the argument runs just as well in a natural language. (I will give a natural language version in a moment.) Next, for a natural language, and for the same reasons that one cannot but suppose that the truth predicate of that language is represented in the language, one cannot but suppose that the denotation relation is represented. After all, 'the denotation of' is a phrase of English. How it manages to achieve its sense is an interesting question. But that it has the sense it has is not in question. And hence, just as is the case with Tarski's Theorem, we cannot accept the result as a *reductio* of this assumption.

4. A PARACONSISTENT INTERPRETATION?

Let us turn, now, to the differences. Tarski's Theorem demonstrates that a certain sentence – one employing the notion of truth – is contradictory. HB's result is a theorem to the effect that *any* functional term of the language has a fixed point. This is not contradictory as such. But it is inconsistent with other truths of arithmetic, e.g., that the successor function has no fixed point. Hence, again, it would seem to follow that (full) arithmetic is inconsistent. In the case of Tarski's Theorem the inconsistency involved concerns the truth predicate. In HB's case, the inconsistency is somewhat different. It could be about the notion of denotation (which would be the strict analogue), if, for example, we took the term $t(x)$ in the Theorem to involve the function δ; but it need not. We can take $t(x)$ to be a purely arithmetic term (i.e., involving purely arithmetic operations; no semantic ones). Hence, it would seem to follow that pure arithmetic (i.e., arithmetic without any semantic notions) is inconsistent.

Should we, then, accept this? The similarities between Hilbert and Bernays' Paradox and the Liar Paradox suggest the possibility of countenancing a dialetheic solution. Dialetheic solutions to the paradoxes of self-reference do not normally implicate ordinary arithmetic in contradiction. A paraconsistent arithmetic is possible, however, and even has something to be said for it.[7] It is interesting to note, then, that a version of the fixed point theorem can be accommodated in an inconsistent but non-trivial arithmetic.

THEOREM. *Provided we restrict the language of first order arithmetic to containing just the function symbols for successor, addition and multiplication (as is usually the case), there is a non-trivial model of complete arithmetic in which every functional term has a fixed point.*

Proof. The model is N_n as defined in Appendix 1 of (G. Priest, 1994). In this model n is an inconsistent object that is identical with (and different from) every greater number. We show that for every functional term of the language with one free variable, $t(x)$, either n is a fixed point or $t(x)$ denotes a constant function (and so has a fixed point). The term $t(x)$ must be built up from the variable x by means of $+$, $\times$, and $'$ (successor). Hence the result may be proved by recursion, with these defining the recursion cases. The basis is simple: the term x has n (and everything else) as a fixed point.

Suppose that the result is true for $t(x)$ and $s(x)$:

(i) If $t(n) = n$ then $t(n)' = n' = n$, so n is a fixed point for $t(x)'$. If, on the other hand, for some i and every x, $t(x) = i$ then $t(x)' = i'$ and so $t(x)'$ denotes the constant function with value $i + 1$.

(ii) Next, consider $t(x) + s(x)$. If n is a fixed point for either of these, say $t(x)$, then $t(n) + s(n) = n + s(n) = n$, so n is a fixed point for the sum. If, on the other hand, $s(x)$ and $t(x)$ denote constant functions, then for some i and j, and for all x, $s(x) = i$, $t(x) = j$, and so $s(x) + t(x) = i + j$, and the sum denotes the constant function with value $i + j$.

(iii) Finally, consider $s(x) \times t(x)$. If n is a fixed point of both, then $s(n) \times t(n) = n^2 = n$, so n is a fixed point of the product. Suppose, on the other hand, that one of these, say $s(x)$, denotes a constant function. Either its value is 0, in which case $s(x) \times t(x) = 0 \times t(x) = 0$, and so the product denotes a constant function. Or else it is non-zero, say $m + 1$; in which case $s(x) \times t(x) = (m + 1) \times t(x)$. Now, $t(x)$ is either a constant function or n is a fixed point. If it is a constant function with value k, then $s(x) \times t(x) = (m + 1) \times k$, and so the product is a constant function with value $(m + 1) \times k$. If n is a fixed point, then $s(n) \times t(n) = (m + 1) \times n = n$. Hence, n is a fixed point for the product.[8] □

5. PARACONSISTENCY REJECTED

Despite the above result, simple paraconsistency does not provide a solution to the problem. The model of the previous section works because the functions represented by the function symbols in question are unbounded above. That is, for any number, k, we can find a number greater than k such that the value of the function applied to it is also greater than k. In such cases, the least inconsistent number can act as a fixed point. This is not the case for bounded functions. And, in fact, it is not difficult to find such functions, the application of the Theorem to which is unacceptable, even by paraconsistent standards.

Consider, for example, the parity function, which maps even numbers to 1 and odd numbers to 0. (This is obviously primitive recursive.) Let it be represented by the function symbol f. Then we have the following:

$$\forall x(fx = 0 \lor fx = 1) \tag{1}$$

$$f0 = 1 \tag{2}$$

$$f1 = 0. \tag{3}$$

By the fixed point theorem, there is a term, s, such that $s = fs$. By (1) and instantiation, $fs = 0 \vee fs = 1$. Argue by dilemma. Suppose that $fs = 0$. Then by si, $s = 0$. But then $fs = 1$ by (2) and si. Hence by si, $1 = 0$. The other horn of the dilemma is symmetrical, using (3) instead of (2). Hence $0 = 1$; and this consequence is acceptable to nobody.[9]

6. DENOTATION

What, then, is to be learned from HB's "Theorem"? We cannot accept it as a "straight" theorem: it has to be a *reductio* of something. This cannot be an assumption of the inexpressibility of denotation (at least in English). The only thing left seems to be the assumption that denotation is a function – or at least, a total function. Some terms must fail to refer. The primitive recursive functions of arithmetic are, of course, total; hence, as far as the denotations of terms that contain only function symbols denoting these goes, there is no worry. However, the fixed point term is, as I observed, the diagonalisation of $t(\delta gx)$, and one cannot assume that this denotes unless denotation is itself a total function, and so beg the question. After all, in informal (natural language) terms, HB's "paradox" is about a term of the form: the successor of the denotation of this term. Call this t. If it denotes, it denotes the successor of t, i.e., of its own denotation. It is natural, then, to suppose that it has no denotation.

Is this observation sufficient to avoid the unacceptable consequences? To answer this, we need to suppose that δ no longer represents a total function, but may represent a partial function. Its semantics can no longer be that of classical first order logic, of course. We must suppose that its semantics, and corresponding logic, is that of some free logic.[10] We need not go into details of these here, for the crucial thing is that if denotation is only a partial function the denotation principle no longer seems right. (If s does not denote, it would seem perverse to insist that 's' denotes s.) This is sufficient to invalidate the proof of the fixed point theorem (though not the lemma on which it depends).

Although we can no longer accept the denotation principle, a restricted form of it is acceptable. Specifically, we can express the fact that y denotes by the condition $\exists x \delta y = x$ (quantifiers, note, do not have to be existentially loaded). Let us write this as Dy. Then for any closed term,

t, $D\langle t\rangle \to \delta\langle t\rangle = t$ is perfectly acceptable. Using this and the lemma, we establish in the obvious way that for any term, $t(x)$, there is a closed term, s, such that $D\langle s\rangle \to s = t(s)$.

Can the damaging implications of the fixed point theorem be extracted from this restricted version of it? The relevant question is obviously whether the fixed point term can be shown to denote, i.e., satisfy D. The term is, recall, the diagonalisation of the term $t(\delta gx)$. In the damaging cases t is simply a monadic function symbol, f. Hence the fixed point term is $f\delta g\langle f\delta gx\rangle$. Now it is natural to assume the Fregean principle that if the argument of a term fails to denote, so does the term itself (that is, that functions are *strict*, in Scott's terminology). The argument of f here is the term $\delta g\langle f\delta gx\rangle$. If this refers to anything, it refers to the denotation of the diagonalisation of the term $f\delta gx$. This is just s itself. Hence the fixed point term s cannot be shown to denote without assuming it to denote. One might use the fact that the fixed point term is ungrounded (in pretty much the sense of Kripke) as an argument for the fact that it lacks a denotation. But in any case, there would seem to be no non-question-begging way of establishing that $D\langle s\rangle$, and so damaging consequences are not forthcoming.

The Fregean principle is not, of course, compulsory (though I think it correct here). However, if one rejects it, the trouble is likely to be avoided for other reasons. If sentences containing non-denoting terms can have a referent (truth value) even though the terms they contain have none, it is natural for the substitutivity of identicals to break down. If, for example, truth values are assigned to atomic sentences containing non-denoting terms by fiat, then it may well happen that $a = b$ and Pa are assigned the value *true*, whilst Pb is assigned the value *false*. The law of substitutivity is applied in the proof of the fixed point theorem, as well as the unpalatable application.[11] Hence, again, the argument fails.

It is worth asking what happens to the argument if the language is enriched with some description operator, say an indefinite description operator, ε. If this is done, function symbols can be omitted from the language, and all appropriate terms can be formed by description. (Denotation is now expressed in the form of a two place predicate, Δxy, satisfying certain conditions, and δx is simply $\varepsilon y\Delta xy$.) What then happens to the argument? The answer depends on the account of descriptions employed, though in principle it is similar. If one assumes that all descriptions denote (as in, for example, the standard semantics of Hilbert's ε-operator) then the fixed point theorem and its unacceptable consequences follow as before. If one (more reasonably) rejects this assumption, then the pertinent question is whether the Fregean principle is adopted. If it is,[12] then the argument would seem to fail since there is no way that the fixed point term can be shown to denote. If it is not, then the substitutivity of identicals is liable to fail,[13] and so the argument again fails. Conceivably, one could have a theory of descriptions that violated the Fregean principle and yet validated substitutivity. However, the present result would itself seem to speak against the correctness of such a theory.

7. A STRENGTHENED ARGUMENT

Adding descriptions to our original machinery does have a notable consequence, however. It is sometimes suggested that a solution to the Liar Paradox is to be found by supposing the liar sentence to suffer from a case of Fregean reference failure: the liar sentence is neither true nor false. A major problem with this suggestion is that it allows the formulation of a slightly different version of the paradox, the extended (or strengthened) Liar, which proceeds by way of "gap-plugging".[14] The solution to Hilbert and Bernays' paradox is also an attempted solution by way of reference failure. It may therefore be wondered whether there is an extended paradox employing gap-plugging in the offing. There is; and the use of descriptions allows us to formulate this.

Let us suppose that ε is an indefinite description operator satisfying the condition $\exists x\varphi(x) \to \varphi(\varepsilon x\varphi(x))$, though we make no assumptions about the behaviour of the term $\varepsilon x\varphi(x)$, when nothing satisfies $\varphi(x)$. Let us then define $\eta x\varphi(x)$, as:

$$\varepsilon y((\exists x\varphi(x) \wedge \varphi(y)) \vee (\neg\exists x\varphi(x) \wedge y = 0)).$$

Since $\exists x\varphi(x) \vee \neg\exists x\varphi(x)$, it is easy to show that:

$$\exists y((\exists x\varphi(x) \wedge \varphi(y)) \vee (\neg\exists x\varphi(x) \wedge y = 0)).$$

Now an uncontentious principle concerning descriptions is the following:

$$\exists y\psi(y) \to D\langle\varepsilon y\psi(y)\rangle.$$

Applying this, it follows that every η-term denotes. Let us call a term *appropriate* if it is either an η-term or it is obtained by applying an arithmetic function symbol to an appropriate term. It follows that all appropriate terms denote. In what follows, all the terms employed are appropriate, and so worries about non-denotation can be set aside.

Now suppose that we define $\delta' y$ as $\eta x(\delta y = x)$. By the lemma of the fixed point theorem, we know that for any arithmetic function symbol,

f, there is an appropriate term, s, such that $s = f\delta'\langle s\rangle$. If we could establish that $\delta'\langle s\rangle = s$, we would therefore have the fixed point theorem as before, in a form that gives unacceptable consequences. Spelled out, this equation is: $\eta x(\delta\langle s\rangle = x) = s$, the left hand side of which is:

$$\varepsilon y((D\langle s\rangle \wedge \delta\langle s\rangle = y) \vee (\neg D\langle s\rangle \wedge y = 0)).$$

Call this term r. By the description principle, we know that:

$$(D\langle s\rangle \wedge \delta\langle s\rangle = r) \vee (\neg D\langle s\rangle \wedge r = 0).$$

It is now tempting to reasons as follows. Since $D\langle s\rangle$ we can rule out the second disjunct.(*) Hence $\delta\langle s\rangle = r$. But by the restricted denotation principle $\delta\langle s\rangle = s$ (since $D\langle s\rangle$). Hence, $r = s$, as required.

The reasoning is classically unimpeachable, and so the problem reappears. However, this is not so for a dialetheist. For the step of the reasoning I marked (*) employs the disjunctive syllogism. (First, note that $D\langle s\rangle \rightarrow \neg(\neg D\langle s\rangle \wedge r = 0)$; then apply the syllogism.) Now this fails in paraconsistent logic.[15] Specifically, if $D\langle s\rangle$ is both true and false, so may $\neg D\langle s\rangle \wedge r = 0$ be, in which case the inference is not truth-preserving.

Accepting that s both does and does not denote in the relevant cases is, in fact, quite plausible. It is a curiously self-referential term of a kind that might be expected to behave strangely. It denotes, as we have seen. However, when f is the successor function, s refers to something that is its own successor, and there is no such thing (or if you think that arithmetic is inconsistent, take f to represent the parity function). Hence, the solution is not at all *ad hoc*.

It may, of course, be that there is some other way of extracting dialetheically unpalatable conclusions from the situation. Only a suitable nontriviality proof for an appropriate paraconsistent theory (arithmetic + denotation + descriptions) would show that this is not the case. At present, such a proof is an interesting open problem.[16]

8. CONCLUSION

We have reviewed Hilbert and Bernays' result and seen that it cannot be allowed to stand, even if one is of a paraconsistent persuasion. The conclusion that they drew from it cannot be allowed to stand either, at least as a claim about natural language. The most plausible conclusion to draw for the result seems to be one concerning denotation: specifically, that not all terms denote. This is not entirely without philosophical import. For the view that all terms denote is a common (though not essential) one in many versions of Meinongianism. We have also seen that this is not sufficient to save classical logicians from the unpalatable consequences of the theorem. There are, notoriously, many different suggestions concerning how truth behaves which are supposed to avoid the Liar Paradox. Most of these have analogues for denotation, and it might be suggested that one of them would avoid the paradox in a satisfactory way. I see no reason to suppose that such suggestions would be more successful at this task than they were at the original one. (They all face well known objections, which seem to me to be insuperable.)[17] At any rate, I leave it to those who would make such suggestions to investigate the possibility. It suffices for me that dialetheism can avoid the problem in a simple and natural way.[18]

9. POSTSCRIPT

There is a close connection between Hilbert and Bernays' result and recursion theory. If we restrict ourselves just to recursive functions, then gödel codes can be taken as names, and decoding provides, in effect, a denotation function. Hilbert and Bernays' result then becomes a version of the Recursion Theorem.[19]

It is interesting to note the response of one of the founding fathers of recursion theory, Kleene, to the recursion theoretic situation,[20] and see whether this can be used to solve the Hilbert and Bernays' paradox. Kleene moved from a theory of total functions to a theory of partial functions expressed in what is, in effect, a free logic based on an underlying 3-valued logic (now called 'strong Kleene'). In this, an identity statement takes the value u (undefined) if either side fails to denote. The original problem is then solved as in Section 6, since the Denotation Principle is not, in general, valid. The extended paradox is also avoided, since the law of excluded middle, on which the proof that all η-terms denote, depends. However, as a solution to the natural-language problem, this is inadequate. $\exists x \delta\langle s\rangle = x$ cannot represent the claim that s denotes. For if s fails to denote $\neg\exists x \delta\langle s\rangle = x$ is undefined, not true.

Kleene introduces a notion of strong identity, $\simeq$, such that $s \simeq t$ is true if s and t denote and have the same denotation, or are both undefined; otherwise it is false. (It is therefore a 2-valued predicate.) If we formulate Hilbert and Bernays' argument in terms of strong identity, we have both the substitutivity of identicals and the Denotation Principle. (If s fails to denote, so does $\delta\langle s\rangle$; hence the identity is true.) We can therefore establish, e.g., that $s \simeq s+1$. This is unproblematic if s has no denotation. We cannot, however, infer the damaging $\exists x\, x \simeq x+1$. Moreover, because

$\simeq$ is a total predicate $\exists x \delta\langle s\rangle \simeq x$ is an adequate representation of the claim that s denotes. But now $\exists x \delta\langle s\rangle \simeq x \vee \neg\exists x \delta\langle s\rangle \simeq x$ is true. Hence we can establish that every δ' term (though not every η-term) denotes, and the extended paradox gives us that $\exists x\ x \simeq x+1$. The two notions of identity therefore demonstrate the familiar trade-off between expressive completeness and inconsistency.[21]

NOTES

[1] For example, the detailed proof of Gödel's second incompleteness theorem. The fact that the book has still not been translated into English is a particularly sad one.

[2] The relevant material can be found in (D. Hilbert and P. Bernays, 1939, pp. 263–78).

[3] See, e.g., (G. Priest, 1987).

[4] See, e.g., (G. Boolos and R. Jeffrey, 1974, p. 176). I will use brackets to indicate the variables free in a formula or term, but not for indicating the arguments of a predicate or function symbol.

[5] First order arithmetic is normally formulated with function symbols for only successor, addition and multiplication. As is well know, however, it can be extended conservatively with function symbols for all primitive recursive functions. Alternatively, it can be formulated with a description operator added, as in Section 6 of this paper. This can be used to provide appropriate terms.

[6] Hilbert and Bernays prove only that the successor function has a fixed point. The observation that the proof generalises to all function symbols is due to Uwe Petersen, who also drew my attention to Hilbert and Bernays' result.

[7] See (G. Priest, 1994). The argument of Appendix 2 of that paper is, in effect, a special case of HB's Theorem. I now think that this argument is incorrect, for reasons that I will give in Section 7 of this paper.

[8] The result also holds if we add a function symbol representing the diagonal function to the language. Let m be any number greater than n which is the code of a term. Let i be the code of its diagonalisation. Under standard gödel codings i is greater than m, and so n. Hence $\delta n = n$ is true in the model.

[9] Alternatively, let f represent the function that maps 0 to 1 and everything else to 0, to obtain a similar result.

[10] Details of various free logics can be found in (E. Bencivenga, 1986).

[11] It is also employed in the proof of the lemma on which the theorem depends. However, in this case the terms in question are ones that denote, and so there is no problem with it.

[12] As it is in the account of descriptions in (T. Smiley, 1960), for example.

[13] As it does, for example, in the semantics for description of (G. Priest, 1979).

[14] See, e.g., (G. Priest, 1987, 1.3).

[15] The $\rightarrow$'s in the argument may be taken as some non-material conditional. Hence the various applications of *modus ponens* are not the disjunctive syllogism.

[16] A partial solution can be found in (G. Priest, 1997).

[17] See (G. Priest, 1987, Chapter 1).

[18] This paper was read at a meeting of the Australasian Association for Logic, University of Otago, August 1994.

[19] As Jon Barwise pointed out to me.

[20] See (S.C. Kleene, 1952, Chapter 12).

[21] See, e.g., (G. Priest, 1987, 1.7).

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